Week 2

ℓ^p- space $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Defn If $p \in [1, \infty)$, define

$$\ell^p = \{\vec{x} = (x_1, x_2, x_3, \cdots) : x_j \in \overset{\text{or } \mathbb{R}}{\overset{\downarrow}{\mathbb{C}}},\ \sum_{j=1}^{\infty} |x_j|^p < \infty\}$$

$$\|\vec{x}\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}}$$

If $p = \infty$, define

$$\ell^\infty = \{\vec{x} = (x_1, x_2, x_3 \cdots) : x_j \in \overset{\text{or } \mathbb{R}}{\overset{\downarrow}{\mathbb{C}}},\ \sup_j |x_j| < \infty\}$$

$$\|x\|_\infty = \sup_j |x_j|$$

We will show $(\ell^p, \|\ \|_p)$ and

$(\ell^\infty, \|\ \|_\infty)$ are normed space

eg 1

$$x = (\overset{x_1}{\overset{\swarrow}{1}}, \overset{x_2}{\overset{\swarrow}{-1}}, 1, -1, \cdots) \quad x_j = (-1)^{j+1}$$

$$|x_j| = 1 \Rightarrow \sup_j |x_j| = 1 < \infty$$

$$\Rightarrow x \in \ell^\infty$$

For $p \in [1, \infty)$

$$\sum_{j=1}^{\infty} |x_j|^p = \sum_{j=1}^{\infty} 1 = \infty$$

$$\Rightarrow x \notin \ell^p$$

eg 2 $0 < r < 1$

$$x = (1, r, r^2, r^3, \cdots) \quad x_j = r^{j-1}$$

Then $\sup_j |x_j| = 1 < \infty$

$$\Rightarrow x \in \ell^\infty$$

For $p \in [1,\infty)$

$$\sum_{j=1}^{\infty} |x_j|^p = \sum_{j=1}^{\infty} r^{(j-1)p} = \frac{1}{1-r^p} < \infty$$

$$\Rightarrow x \in \ell^p$$

eg 3

$$x = (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \cdots) \quad x_j = \frac{1}{j}$$

$$|x_j| = \frac{1}{j} \Rightarrow \sup_j |x_j| = 1 < \infty$$

$$\Rightarrow x \in \ell^\infty$$

For $p = 1$

$$\sum_{j=1}^{2^n} |x_j| = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n}$$

first 2^n terms

$$= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}$$

$$+ \overbrace{\frac{1}{9} + \cdots + \frac{1}{16}} + \cdots + \overbrace{\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}}$$

$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \cdots + \frac{1}{8}}$$

$$+ \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}} + \cdots + \underbrace{\frac{1}{2^n} + \cdots + \frac{1}{2^n}}$$

$$= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n}$$

$$= 1 + \frac{n}{2}$$

i.e. $\displaystyle\sum_{j=1}^{2^n} |x_j| \geq 1 + \frac{n}{2}$

$n \to \infty$

$$\Rightarrow \sum_{j=1}^{\infty} |x_j| = \infty$$

$$\Rightarrow x \notin \ell^1$$

For $1<p<\infty$,

$$\sum_{j=1}^{n} |x_j|^p = \sum_{j=1}^{n} \frac{1}{j^p}$$

$$= 1 + \sum_{j=2}^{n} \frac{1}{j^p}$$

$$< 1 + \int_1^n \frac{1}{t^p}\,dt$$

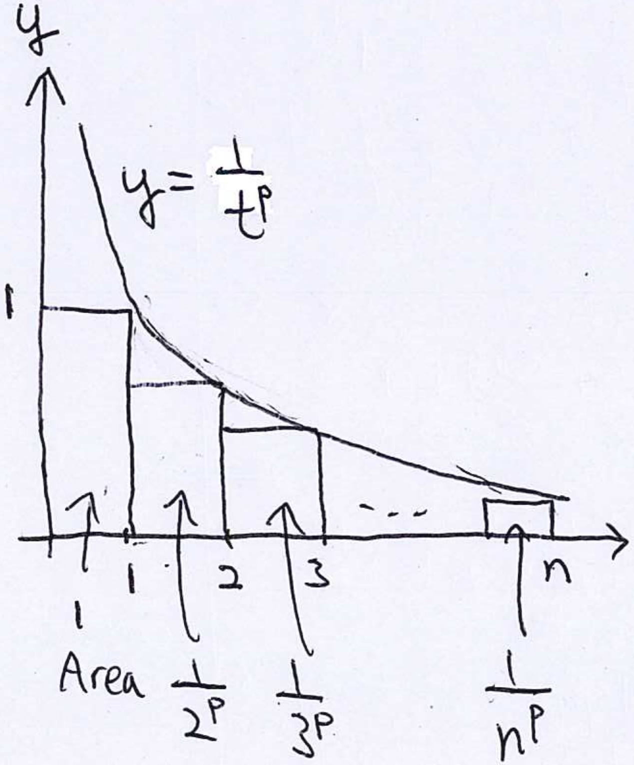
$$= 1 + \int_1^n t^{-p}\,dt$$

$$= 1 + \left[\frac{t^{1-p}}{1-p}\right]_1^n$$

$$= 1 + \frac{1}{1-p}\left(n^{1-p} - 1\right)$$

$$= 1 + \frac{1}{p-1}\left(1 - n^{1-p}\right)$$

$$< 1 + \frac{1}{p-1}$$



Take $n\to\infty$

$$\Rightarrow \sum_{j=1}^{\infty} |x_j|^p \le 1 + \frac{1}{p-1} < \infty$$

$$\Rightarrow x \in \ell^p$$

<u>Q</u> What happens to this proof if $p=1$

Reason:

$$\int \frac{1}{t}\,dt = \ln|t|$$

<u>Rmk</u> For $1<p<q<\infty$

$$\ell^1 \subsetneq \ell^p \subsetneq \ell^q \subsetneq \ell^\infty$$

$\subsetneq$ proper subset

<u>Q</u> Find x s.t.

$x \in \ell^q$ and $x \notin \ell^p$

Q1 Is ℓ^p a vector space?, $1 \le p \le \infty$

Q2 If so, does $\|x\|_p$ define a norm?

For Q1, note

- $\ell^p \underset{\text{subset}}{\subset}$ vector space of all sequences
- $\vec{0} = (0, 0, 0, \cdots) \in \ell^p$
- If $x \in \ell^p$, $\alpha \in \mathbb{C}$ (or $\mathbb{R}$)

then $\sum_{j=1}^{\infty} |x_j|^p < \infty$

then $\sum_{j=1}^{\infty} |\alpha x_j|^p < \infty$

$\Rightarrow \alpha x \in \ell^p$

$p \neq \infty$

If $p = \infty$, similar

$\Rightarrow \ell^p$ is closed under scalar multiplication

· More difficult part:

To show ℓ^p is closed under vector addition

For Q2,

the difficult part is to show

$$\|x + y\|_p \le \|x\|_p + \|y\|_p$$

To do these, need a few inequalities

Important inequalities

① Young's inequality

Let $p, q > 1$, $a, b \geq 0$

and $\dfrac{1}{p} + \dfrac{1}{q} = 1$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Pf **Case 1** If $a = 0$ or $b = 0$

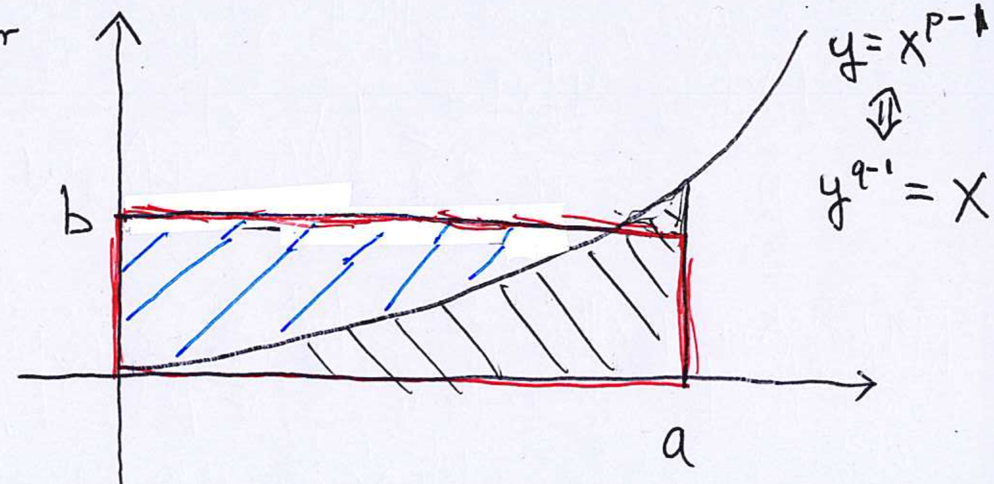
Then L.H.S. $= 0 \leq$ RH.S

Case 2 If $a \neq 0$, $b \neq 0$

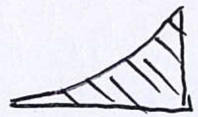
$$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{p+q}{pq} = 1 \Rightarrow p + q = pq$$

$$pq - p - q + 1 = 1 \Rightarrow (p-1)(q-1) = 1$$

Consider



Compare areas:

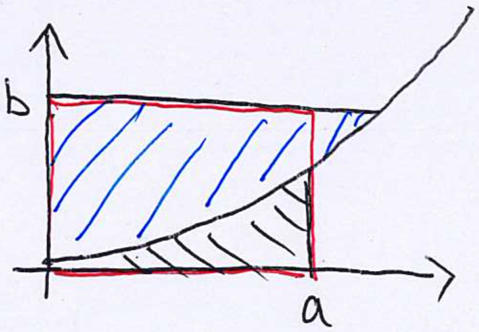
$ab \leq$  $+$ 

$$= \int_0^b y^{q-1}\,dy + \int_0^a x^{p-1}\,dx$$

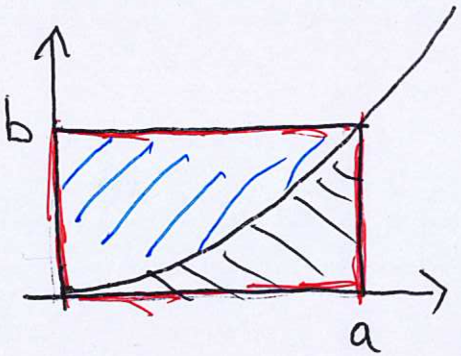
$$= \frac{1}{q}\left[y^q\right]_0^b + \frac{1}{p}\left[x^p\right]_0^a$$

$$= \frac{b^q}{q} + \frac{a^p}{p}$$

Other cases:



Similar

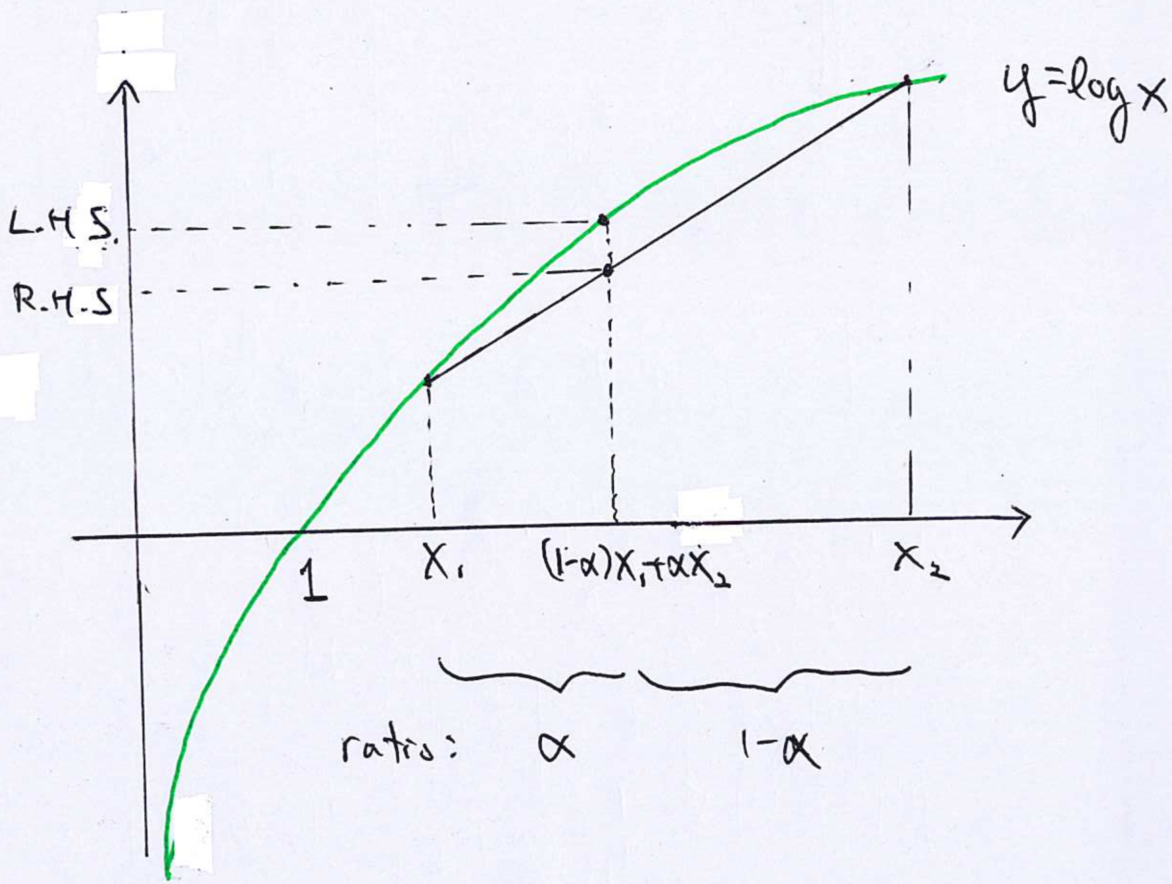


Equality case $\Leftrightarrow a^p = b^q$

$$\left(\begin{array}{lll} b = a^{p-1} & \Leftrightarrow & b^{q-1} = a \\ ab = a^p & & b^q = ab \end{array}\right)$$

Another pf of Young's inequality is by using the concavity of log function (HW)

$$\log((1-\alpha)x_1 + \alpha x_2) \geq (1-\alpha)\log x_1 + \alpha \log x_2 \quad \text{for } \alpha \in (0,1)$$



2 Hölder inequality

Let $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$

Let $x \in \ell^p$, $y \in \ell^q$, then

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \underbrace{\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}}_{\|x\|_p} \underbrace{\left(\sum_{j=1}^{\infty} |y_j|^q\right)^{\frac{1}{q}}}_{\|y\|_q}$$

Pf Case 1 $\|x\|_p = 0$ or $\|y\|_q = 0$

Then $x = \vec{0} = (0, 0, 0, \cdots)$

or $y = \vec{0} = (0, 0, 0, \cdots)$

$\Rightarrow$ L.H.S. $= 0 \leq$ R.H.S. $= 0$

The inequality is true

Case 2 $\|x\|_p = \|y\|_q = 1$

Young

$$\Rightarrow |x_j y_j| \leq \frac{|x_j|^p}{p} + \frac{|y_j|^q}{q}$$

$$\Rightarrow \sum_{j=1}^{\infty} |x_j y_j| \leq \frac{1}{p} \underbrace{\sum_{j=1}^{\infty} |x_j|^p}_{(\|x\|_p)^p} + \frac{1}{q} \underbrace{\sum_{j=1}^{\infty} |y_j|^q}_{(\|y\|_q)^q}$$

L.H.S.

$$= \frac{1}{p} + \frac{1}{q} = 1$$

R.H.S. $= \|x\|_p \|y\|_q = (1)(1) = 1$

$\Rightarrow$ The inequality is true

Case 3 $\|x\|_p, \|y\|_q \neq 0$

Let $\tilde{x}_j = \dfrac{x_j}{\|x\|_p}$ $\tilde{y}_j = \dfrac{y_j}{\|y\|_q}$

$$\|\tilde{x}\|_p = \left(\sum_{j=1}^{\infty} |\tilde{x}_j|^p\right)^{\frac{1}{p}}$$

$$= \left[\sum_{j=1}^{\infty}\left(\frac{|x_j|}{\|x\|_p}\right)^p\right]^{\frac{1}{p}}$$

$$= \frac{1}{\|x\|_p}\left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}}$$

$$= \frac{1}{\|x\|_p}\|x\|_p$$

$$= 1$$

Similarly, $\|\tilde{y}\|_q = 1$

Case 2

$$\Rightarrow \sum_{j=1}^{\infty} |\tilde{x}_j \tilde{y}_j| \leq 1$$

$$\Rightarrow \sum_{j=1}^{\infty} \frac{|x_j y_j|}{\|x\|_p \|y\|_q} \leq 1$$

$$\Rightarrow \sum_{j=1}^{\infty} |x_j y_j| \leq \|x\|_p \|y\|_q$$

$\Rightarrow$ Hölder's inequality

3 Minkowski inequality

Let $p \geq 1$, $x = (x_i), y = (y_j) \in \ell^p$

Then

$$\left(\sum_{j=1}^{\infty} |x_j + y_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} |y_j|^p\right)^{\frac{1}{p}}$$

$\uparrow$ assumption $\Rightarrow$ finite $\qquad \uparrow$ finite

Pf Case 1 : $p = 1$

$$|x_j + y_j| \leq |x_j| + |y_j|$$

$\swarrow$ Δ inequality for real numbers

$$\Rightarrow \sum_{j=1}^{\infty} |x_j + y_j| \leq \sum_{j=1}^{\infty} |x_j| + |y_j|$$

$$= \sum_{j=1}^{\infty} |x_j| + \sum_{j=1}^{\infty} |y_j|$$

$\Rightarrow$ Minkowski for $p = 1$

Case 2 : $p > 1$. Let $\frac{1}{p} + \frac{1}{q} = 1$

$$|x_j + y_j|^p = |x_j + y_j|^{p-1} |x_j + y_j|$$

$$\leq |x_j + y_j|^{p-1} (|x_j| + |y_j|)$$

$$= |x_j + y_j|^{p-1} |x_j| + |x_j + y_j|^{p-1} |y_j|$$

(a) $$\sum_{j=1}^{n} |x_j + y_j|^p \leq \left(\sum_{j=1}^{n} |x_j + y_j|^{p-1} |x_j|\right) + \left(\sum_{j=1}^{n} |x_j + y_j|^{p-1} |y_j|\right)$$

$$\sum_{j=1}^{n} |x_j + y_j|^{p-1} |x_j|$$

$pq = p + q$

$(p-1)q = p$

$$\leq \left(\sum_{j=1}^{n} |x_j + y_j|^{(p-1)q}\right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$$

(b) $$= \left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$$

Similarly

$$\text{(c)} \quad \sum_{j=1}^{n} |x_j+y_j|^{p-1}|y_j|$$

$$\leq \left(\sum_{j=1}^{n} |x_j+y_j|^p\right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}$$

(a) + (b) + (c)

$$\Rightarrow \sum_{j=1}^{n} |x_j+y_j|^p$$

$$\leq \left(\sum_{j=1}^{n} |x_j+y_j|^p\right)^{\frac{1}{q}} \left[\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}\right]$$

$$\Rightarrow \left(\sum_{j=1}^{n} |x_j+y_j|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}$$

Note $1-\frac{1}{q}=\frac{1}{p}$, take $n\to\infty \Rightarrow$ Minkowski inequality

Rmk

- The finite versions of these inequality are also true
- These inequalities are also true in the case

$$p=1,\ q=\infty \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

or $p=\infty,\ p=1$

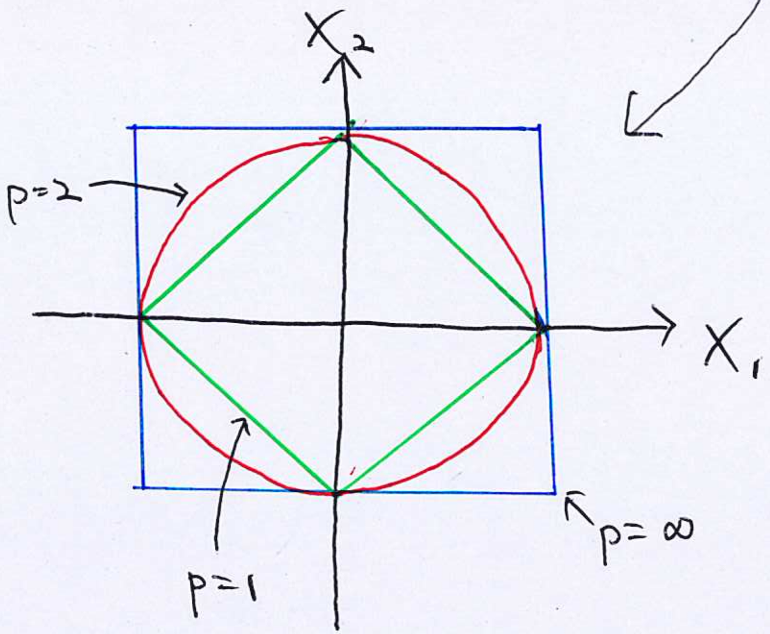
ℓ^p norm in $\mathbb{R}^n$ or $\mathbb{C}^n$

$X = (x_1, x_2, \cdots, x_n)$

$\|x\|_\infty = \sup |x_i|$

$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}}$

<u>Picture</u> $n=2$ $(\mathbb{R}^2)$



What is the unit circle in ℓ^p-norm?

i.e. $\{X \in \mathbb{R}^2 : \|X\|_p = 1\}$

<u>eg</u> $p=1$ $\|x\|_1 = 1$

$|x_1| + |x_2| = 1$

$p=2$ $\|x\|_2 = 1$

$\sqrt{|x_1|^2 + |x_2|^2} = 1$

$p=\infty$ $\|x\|_\infty = 1$

$\sup\{|x_1|, |x_2|\} = 1$